MAT 1322
Lecture 11

15/02/2017

More Series

Recall · integral test for convergence
$\boxed{\cdot \text{ comparison test}}$

We have 2 series $\sum_{i=1}^{\infty} a_i$, $\sum_{i=1}^{\infty} b_i$

If $a_i \le b_i$ ie. $\sum_{i=1}^{\infty} a_i \le \sum_{i=1}^{\infty} b_i$

IF series in question is $\sum_{i=1}^{\infty} a_i$:

case 1:
if $\sum_{i=1}^{\infty} b_i$ converges, so does $\sum_{i=1}^{\infty} a_i$

case 2:
if $\sum_{i=1}^{\infty} b_i$ diverges, we have no conclusion about $\sum_{i=1}^{\infty} a_i$

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Ex Comparison test

$\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ Does it converge or diverge?

IDEA: compare to $\sum_{k=1}^{\infty} \frac{1}{k}$, but $\ln(1)=0$
$\ln(2) < 1$
$\ln(3) > 1$

So, compare from $k=3$ on.

$$\sum_{k=3}^{\infty} \frac{\ln(k)}{k} \geq \sum_{k=3}^{\infty} \frac{1}{k}$$ → This part of the harmonic series, which we know diverges

If $\sum_{k=1}^{\infty} a_k$ converges, so does $\sum_{k=m}^{\infty} a_k$ for any $m \geq 1$

If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=m}^{\infty} a_k$, $m \geq 1$

(in the above example we used $m=3$)

LIMIT COMPARISON

We have a series $\sum_{i=1}^{\infty} a_i$ and we try to determine whether it converges or diverges by comparing it to $\sum_{i=1}^{\infty} b_i$

Ex $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ → looks like geometric series, so we can guess that it converges.

<u>Limit comparison</u> If $\sum_{i=1}^{\infty} a_i, \sum_{i=1}^{\infty} b_i$ then if $\lim_{i\to\infty} \frac{a_i}{b_i} = C \underline{\underline{>}} 0$, then either both converge or diverge.

Ex

Compare to $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{\frac{2^n}{2^n}}{\frac{2^n-1}{2^n}} = \lim_{n\to\infty} \frac{1}{1-\underbrace{2^{-n}}_{\to 0}} = 1 \quad > 0$$

Because $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

SERIES TO USE FOR COMPARISON

- $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ geometric series, converges if $|q|<1$.
- $\sum_{n=1}^{\infty} \frac{1}{n^p}, p>1$ converges
- $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, harmonic series

ESTIMATING SUMS

$\sum_{i=1}^{\infty} a_i$ converges, and we'd like to know its value up to a few digits.

We know that $\lim_{i\to\infty} a_i = 0$, so from some n on we get that $a_m, m>n$ satisfies $|a_m| < \varepsilon$ where ε is tiny (can be $\varepsilon = 10^{-20}$).

Ex $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$ converges (can determine by comparing to $\sum_{n=1}^{\infty} \frac{1}{n^3}$)

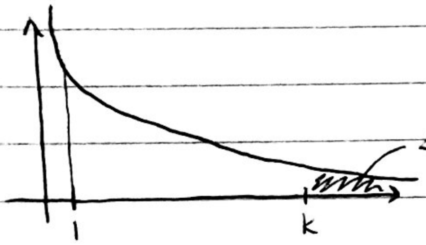
$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_k}_{\text{relevant digits}} + \underbrace{a_{k+1} + a_{k+2} + \dots +}_{\text{remainder}}$$

If the index k is huge, then the remainder becomes tiny despite that it's an infinite sum $\sum_{n=k+1}^{\infty} a_n$ (for $\sum_{n=1}^{\infty} a_n$ converges)

Practical question: how big does k have to be such that the remainder is smaller than a given bound? (eg 10^{-3}, 3 digits)

USE INTEGRALS!

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^3}$. If the remainder of $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is less than 10^{-4}, so is the remainder of $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$ because $1/n^3 > \frac{1}{n^3+3}$



shift k to the right until area is $< 10^{-4}$.

$$\int_k^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left(\int_k^t \frac{1}{x^3}\,dx\right) = \lim_{t\to\infty}\left(\frac{-1}{2}x^{-2}\right)\Big|_k^t$$

$$= \lim_{t\to\infty}\left(\frac{-1}{2}t^{-2} + \frac{1}{2}k^{-2}\right) = \underbrace{\lim_{t\to\infty}\left(\frac{-1}{2}t^{-2}\right)}_{0} + \frac{k^{-2}}{2} = \boxed{\frac{1}{2k^2}}$$

When is this less than 10^{-4}?

$\frac{1}{2k^2} < 10^{-4}$ solve for k!

$1 < 2k^2 \cdot 10^{-4}$

$10^4 < 2k^2$

$\frac{1}{2}10^4 < k^2$

$\frac{1}{\sqrt{2}} 10^2 < k$, so $\boxed{k > \frac{100}{\sqrt{2}}}$

Alternating Series (11.4)

Ex (1) $\sum_{i=1}^{\infty} (-1)^i \cdot \frac{1}{i} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots$

(alternating harmonic series)

↳ converges

(2) $\sum_{i=1}^{\infty} (-1)^i = -1 + 1 - 1 + 1 - 1 + \dots$

One single method to check convergence:

ALTERNATING SERIES TEST

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \dots$$

IF case 1: $a_{n+1} \leq a_n$ for all $n \geq N$

case 2: $\lim_{n\to\infty} a_n = 0$

THEN the series converges.

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, alternating harmonic series

(C1) $a_{n+1} \leq a_n$; here: $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$$\frac{1}{n+1} \leq \frac{1}{n}$$

$$n \leq n+1 \quad ✓$$

(C2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

→ series converges.

The first condition says a_n are decreasing.

Technical helper: look at $f(x)$ s.t. $f(n) = a_n$ (here, $f(x) = \frac{1}{x}$)

→ a_n is decreasing if and only if $f(x)$ is decreasing for $x \geq N$

Use $f'(x)$ to check, $f(x)$ is decreasing if $f'(x) < 0$.

In the example, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, $f(x) = \frac{1}{x}$

So, $f'(x) = \frac{-1}{x^2}$, of course $\frac{-1}{x^2} < 0$ for all $\underline{x \geq 1}$